wall temperature. The $g_w = 0$ case corresponds to $T_w = 294$ K. The velocity profiles for $\beta = 0.5$ are shown in Fig. 2. For comparison, the constant property, $c \equiv 1$, solutions are also presented on the figure as dashed lines.

The shear τ_w and the heat flux q_w at the wall can be determined from the solution by the following expressions

$$\tau_w = (\rho_w \mu_w u_e^2 / \sqrt{2\xi}) f_{\eta\eta}(0)$$

$$q_w = (K_w \rho_w u_e h_{se} / c_{pw} \sqrt{2\xi}) g_\eta(0)$$

The results for the wall shear parameter $f_{\eta\eta}(0)$ and the wall stagnation enthalpy gradient $g_\eta(0)$, which determines the heat flux, are shown in Figs. 3 and 4, respectively, as functions of the pressure gradient parameter β. It can be seen that both quantities vary considerably from their values at constant $\rho\mu (c \equiv 1)$. As is seen in Fig. 4, $g_\eta(0)$ varies by a factor of approximately 5 for $g_w = 0.02$ case. Hence, the assumption of constant $\rho\mu$ in a hot hydrogen boundary layer would cause a significant error in calculating the heat-transfer rate. For engineering applications, the correlation of the parameter $g_\eta(0)$ as a function of the $\rho\mu$ ratio across the boundary-layer can be obtained from Fig. 4. For example, when we consider the stagnation point flow, $\beta = 0.5$, the correlation can be written as

$$g_\eta(0) = g_\eta(0) \mid_{c=1} (\rho_e \mu_e / \rho_w \mu_w)^{0.8}$$

where $g_\eta(0) \mid_{c=1}$ is the corresponding constant-property value from Ref. 11 (Fig. 5). In obtaining the above correlation, the calculations have been done for the range of g_w from 0 to 1.

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Technical Comments

Comment on "Practical Aspect of the Generalized Inverse of a Matrix"

B.E. Gatewood*

Ohio State University, Columbus, Ohio

HASSIG[1] uses additional constraint equations to obtain a unique solution for a system of equations with more unknowns (n) than equations (m). His solution of

$$[A]_{m,n}\{x\}_{n,1} = \{y\}_{m,1}, \qquad m<n, \tag{1}$$

with the constraints

$$\{x\}_{n,1} = [A^T]_{n,m}\{\alpha\}_{m,1}, \qquad m<n, \tag{2}$$

is given as

$$\{x\}_{n,1} = [A^T]_{n,m}[[A]_{m,n}[A^T]_{n,m}]^{-1}\{y\}_{m,1}, \qquad m<n, \tag{3}$$

where $[A]_{m,n}$ has rank m.

Although Hassig's Note[1] is concerned with the generalized inverse form in Eq. (3), there are questions as to the physical meaning of the solution (3) and as to a best solution in certain applications. Since any matrix $[B]_{n,m}$ with rank m could be used in place of $[A^T]_{n,m}$ in Eqs. (2) and (3), what is special about $[A^T]_{n,m}$?

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*Professor, Aeronautical and Astronautical Engineering Dept. Member AIAA.

If Eq. (1) is substituted into in Eq. (3) for $\{y\}$, then

$$[K]_{n,n}\{x\}_{n,1} = \{0\}_{n,1} \tag{4}$$

$$[K]_{n,n} = [[I] - [A^T][[A][A^T]]^{-1}[A]]_{n,n} \tag{5}$$

Since $[K]_{n,n}$ is not zero but has rank $n-m$, Eq. (4) imposes $n-m$ explicit independent constraint equations directly on the x_i. These equations could be combined directly with Eq. (1) to give the $\{x\}$ solution. It is evident that these same $n-m$ equations are also the conditions to make the sum of the squares of the x_i a minimum, or

$$\sum_{i=1}^{n} (x_i)^2 = \text{minimum} \tag{6}$$

a result noted by Greville.[2] Thus, the physical meaning of using $[A^T]_{n,m}$ in Eqs. (2) and (3) is to obtain the set of x_i satisfying Eq. (6). No other $[B]_{n,m}$ matrix in place of $[A^T]_{n,m}$ will do this. However, is this solution (3), restrained by condition (6), a best solution in actual applications?

Consider the example of representing a function over an interval with a set of n known functions with unknown coefficients x_i. If m collocation points ($m<n$) are used to determine the coefficients, Eq. (1) results. The best values for the x_i are those that give the best representation of the function in the interval. In the case of using polynomials to represent the exponential function, the solution (3) is fair but it is easy to select $[B]$ matrices in place of $[A^T]$ that give better results for the x_i.

In redundant structural truss problems, Eq. (1) gives the equilibrium equations and Eq. (2), with a $[B]$ matrix involving geometry and material properities, gives the deflection equations. In one special case $[B]$ is $[A^T]$, but the result is very poor for most practical truss problems.

Thus, although the solution (3) satisfies the conditions (1), (2), (4), and (6), it does not necessarily give the best values for the x_i in applications. It can be used as a starting point for the solution and can be taken as a solution if nothing other than Eq. (1) is known about the system. Obviously, in applications it is desirable to have sufficient known conditions to solve the problem, or $m=n$.

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Reply by Author to B. E. Gatewood

Hermann J. Hassig*
Lockheed-California Company, Burbank, Calif.

PROFESSOR Gatewood's question: "....what is special about $[A^T]_{n,m}$?"[1] is well taken. Writing Eq. (7) of Ref. 2 as

$$\{x\}_{n,l} = [B]_{n,m}\{\alpha\}_{m,l} \qquad m<n \tag{1}$$

in order to put constraints on $\{x\}_{n,l}$ is considered common practice. The purpose of Ref. 2 was to show that a special form of $[B]$, namely $[B]=[A^T]_{n,m}$ leads to an expression for $\{x\}$ identical to the generalized inverse of Ref. 3.

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*Research and Development Engineer.

Comment on "Prediction of Turbulent Boundary Layers at Low Reynolds Numbers"

Roger L. Simpson*
Southern Methodist University Dallas, Texas

THE purpose of this Comment is to point out that there is some confusion about some experimental data[1] that appears to be amplified with each successive paper on low Reynolds number turbulent boundary layers. Pletcher[2] only presented the conclusions of Squire[3] and Baker and Launder[4] about the *blowing* data[1] without examining the basis for these conclusions or Coles[5] reevaluation of these data.

In particular Squire and Baker and Launder both ignored the requirement that for slow variations of the non-dimensional blowing velocity V_w/U_∞ along a porous surface with zero pressure gradient

$$(C_f/2)=f(Re_\theta,\ (V_w/U_\infty)) \tag{1}$$

This statement simply reflects the fact that local conditions largely determine the flow structure. Simpson's analysis[1] of McQuaid's[6] blowing data indicated that this equation was not satisfied by those data. This prompted Squire, who was McQuaid's advisor, to reanalyze Simpson's momentum balance data, even though Simpson had used several different experimental techniques to obtain a "best estimate" of $C_f/2$. Simpon's best estimate $C_f/2$ values[7] for twenty different series of velocity profiles with different variations in V_w/U_∞ satisfied the aforementioned equation. For five series of velocity profiles with a discontinuity in wall blowing,[8] the same equation was satisfied downstream of the relaxation after the discontinuity. Coles also found Simpson's $C_f/2$ estimates to be very reasonable.[1]

Simpon's[9] arguments on the effects of low Reynolds number on zero pressure gradient boundary layers on impermeable surfaces were not strongly dependent on the skin friction coefficient $C_f/2$ as implied by Pletcher, but rather on dimensional analysis. First Simpson noticed that in the outer region of such a low Reynolds number boundary layer ($Re_\theta <$ *6000*)

$$(U/U_\infty)=g(y/\delta) \tag{2}$$

appeared to be closely satisfied. If one requires a law-of-the-wall velocity profile near the wall of the form

$$U/U_\infty (C_f/2)^{1/2} = h\left(\frac{yU_\infty}{v}(C_f/2)^{1/2}\right) \tag{3}$$

then a Millikan argument requires a logarithmic common overlap region. This condition implies that the von Karman "constant" K varies as $(C_f/2)^{1/2}$, *if* there is an overlap region. Since Simpson's skin friction values compared favorably with the accepted relation

$$(C_f/2)=0.0128\,Re_\theta^{-1/4}$$

for low Reynolds numbers, $K \sim Re_\theta^{-1/8}$ for $Re_\theta < 6000$.

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*Professor of Mechanical Engineering. Associate Fellow AIAA.